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Numerical analysis of pressure transients in bubbly two-phase mixtures by explicit-implicit methods

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SUMMARY

The one-dimensional equations for transient two-phase flow are a system of nonlinear hyperbolic partial differential equations, expressible, under certain assumptions, in conservation form. Inasmuch as the use of the method of characteristics becomes complicated if shock waves are present, it is easier to follow a gas-dynamics approach and employ one of the available procedures for solving one-dimensional systems of conservation equations. A recently introduced technique, due to McGuire and Morris [1, see also 12] and known as an Explicit-Implicit method, is used here for a simple boundary-value problem of wave propagation in bubbly two-phase mixtures, and is found to be simple and versatile. A comparison of this method with the well-known Lax–Wendroff (two-step) scheme demonstrates that shock fronts are simulated better, oscillations behind the shocks are smoothable by parameter adjustment, and computation time is reduced when the Explicit-Implicit method is employed.

1. Introduction

The problem of transient analysis of two-phase flows is complicated by the fact that the evaluation of the characteristic and the compatibility equations is often difficult, as illustrated by Prosperetti and Wijngaarden [2] and Lyczokowski *et al.* [3]. Moreover, the possibility of the formation of shock waves due to the steepening of compression waves usually limits the use of the method of characteristics because of the need to incorporate the shock equations as internal boundary conditions in the characteristic grid. If the governing field equations of the problem can be expressed in conservation form, it is often easier to follow the approach used in gas dynamics, wherein discontinuities, such as shocks, are permitted. The well-known Lax–Wendroff scheme (see Ames [11, 12]) has been successfully applied by Kranenburg [4] for problems of transient cavitation in pipe lines. Martin *et al.* [5] have also used the Lax–Wendroff scheme for the numerical integration of conservation equations based on a homogeneous model for the analysis of transients in air-water mixtures. For a separated-flow model, based on certain reasonable assumptions, Martin and Padmanabhan [6] have shown that the governing equations are expressible in conservation form whereupon the Lax–Wendroff numerical scheme can be applied.

For wave propagation in two-phase conduit flow the existence of a pressure gradient results in a variation of the wave propagation speed along the conduit, even for steady flow. In a computational grid the length and time steps, Δx and Δt, have to satisfy the Courant–Friedrichs–Lewy (CFL) condition in order to assure stability of explicit schemes, as

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discussed by Lax and Wendroff [7]. These schemes are most accurate if $\Delta x/\Delta t$ is equal to the local celerity of the disturbance in the $x-t$ space. However, for obvious reasons this is seldom possible in transient computations of two-phase flows. Consequently, wave distortion occurs, in effect changing the form of the disturbance by causing an overshooting of the wave front, which is followed by high-frequency oscillations. One method for suppressing these irregularities is the introduction of artificial viscosity terms. Vliegenthart [8] proposed the use of a Shuman filtering operator, which is claimed to be able to remove nonlinear instabilities under certain conditions. For transient two-phase conduit flow, Kranenburg [4] and Martin *et al.* [5] used this method for suppressing oscillations and overshooting.

By combining a two-step explicit scheme [9] and an implicit scheme of second-order accuracy [10], McGuire and Morris [1] introduced an attractive and simple Explicit-Implicit method. The overshooting and attendant high-frequency oscillations were reduced by the proper selection of two parameters used in the procedure. The method appears to be particularly applicable for boundary-value problems since at each step the implicit set of calculations can incorporate any changes in the boundary at the step. However, some of the computations along the boundary may have to be carried out using the method of characteristics.

The objective of this paper is to illustrate the use of the Explicit-Implicit method, and to compare it with the Lax–Wendroff two-step scheme for a boundary-value problem of the transient analysis of two-phase flow of a flowing bubbly mixture. The ends of the conduit are represented by a constant pressure reservoir at one end and a quick-acting valve at the other. The transient is generated by a rapid closure of the valve at the downstream end of the pipe. Due to pipe friction an initial steady-flow pressure gradient exists in the horizontal pipe. As the mesh ratio, $p = \Delta t/\Delta x$, for the finite-difference scheme employed is of the order of 10^{-3} because of the stability requirement of Courant–Friedrich–Lewy, this paper also serves as a verification of the use of the Explicit-Implicit method for computations with extremely small mesh ratios. It should be noted that the results reported by McGuire and Morris [1] correspond to much higher values of p, 0.25 to 1.

2. Conservation equations for homogeneous model

Martin *et al.* [5] proposed a homogeneous model for the analysis of transients in bubbly mixtures, which basically consists of one-dimensional equations of conservation of mass for each of the phases, and conservation of momentum for the mixture. The three field equations can be expressed in the conservation form as follows:

Conservation of mass of the liquid phase:

$$\frac{\partial}{\partial t}\left\{(1-\alpha)\left[1+P\left(\frac{1}{K_l}+\frac{D\mu}{Ee}\right)\right]\right\}+\frac{\partial}{\partial x}\left\{(1-\alpha)v\left[1+P\left(\frac{1}{K_l}+\frac{D\mu}{Ee}\right)\right]\right\}=0, \tag{2.1}$$

Conservation of mass of the gas phase:

$$\frac{\partial}{\partial t}(\alpha P^{1/\gamma})+\frac{\partial}{\partial x}(\alpha P^{1/\gamma}v)=0, \tag{2.2}$$

Conservation of momentum of the mixture:

$$\frac{\partial}{\partial t}\left\{(1-\alpha)v\left[1+P\left(\frac{1}{K_l}+\frac{D\mu}{Ee}\right)\right]\right\}$$
$$+\frac{\partial}{\partial x}\left\{(1-\alpha)v^2\left[1+P\left(\frac{1}{K_l}+\frac{D\mu}{Ee}\right)\right]+\frac{P}{\rho_l}\right\}=(1-\alpha)\left[g\sin\theta-\frac{f}{2D}v|v|\right]. \qquad (2.3)$$

In the above equations α represents the average void fraction, P the cross-sectional average pressure, v the velocity of the mixture, K_l the bulk modulus of elasticity of the liquid, D the diameter of the pipe, μ a pipe-constraint factor, e the thickness of the pipe wall, ρ_l the density of liquid, f the Darcy–Weisbach resistance coefficient, γ the polytropic exponent, x the distance along the pipe and t the time. Equations (2.1) to (2.3) are in conservation form and are a system of nonlinear hyperbolic partial differential equations with α, P and v as dependent variables, and x and t as independent variables. Martin *et al.* [5] derived the characteristic roots of the system and developed the compatibility equations.

In this paper the system of equations (2.1) to (2.3) will be numerically integrated in order to compare the Explicit-Implicit scheme with the Lax–Wendroff scheme. The following characteristic and compatibility equations, derived by Martin *et al.* [5], will be used to simulate the boundary conditions.

$$\frac{dx}{dt}=v\pm w, \qquad (2.4)$$

$$\frac{dP}{dt}\pm\frac{w}{c_1}\frac{dv}{dt}\mp\frac{wc_2}{c_1}=0, \qquad (2.5)$$

and

$$\frac{dx}{dt}=v, \qquad (2.6)$$

$$\frac{dP}{dt}+c_3\frac{d\alpha}{dt}=0. \qquad (2.7)$$

where

$$w=\left[(1-\alpha)\rho_l\left(\frac{D\mu}{Ee}+\frac{\alpha}{K_g}+\frac{(1-\alpha)}{K_l}\right)\right]^{-\frac{1}{2}} \qquad (2.8)$$

represents the wave propagation speed in the mixture. The parameters c_1, c_2 and c_3 are defined by

$$c_1=\frac{1}{\rho_l(1-\alpha)}, \qquad (2.9)$$

$$c_2=g\sin\theta-\frac{f}{2D}v|v|, \qquad (2.10)$$

and

$$c_3 = \left[\alpha(1-\alpha)\left(\frac{1}{K_g} - \frac{1}{K_l}\right)\right]^{-1}. \tag{2.11}$$

3. Transient two-phase bubbly mixture in a long conduit

The homogeneous model of two-phase flow described by equations (2.1) to (2.3) is used to simulate the transient response of a flowing bubbly air-water mixture subsequent to the rapid closure of a value at the downstream end of a horizontal pipe 25 mm in diameter and 18 m long. As shown in Figure 1, the pipe is connected to a constant pressure reservoir at one end and a quick acting valve at the other end. In the example to be considered, the constant reservoir pressure P_0 is assumed to be 0.38 *mPa* absolute and the void fraction α_0 at pressure P_0 is taken as 0.014. The average steady-flow velocity $v_0 = 1.5$ m/sec., and the time of valve closure is assumed to be 0.03 sec. For simplicity a linear valve closure is assumed. The pulse propagation velocity with only water in the pipe is assumed to be 600 m/sec.

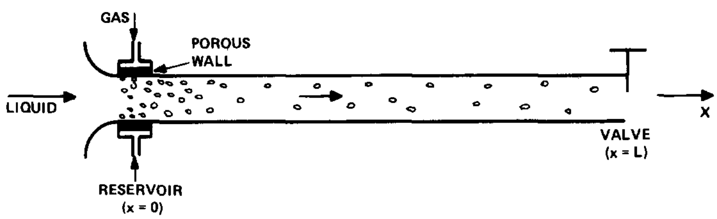

Figure 1. Schematic of flowing two-phase bubbly mixture in conduit.

The boundary conditions are simulated by use of the method of characteristics, equations (2.4–2.7). The value of the time interval is re-computed each time step from the CFL condition, $\Delta t = \Delta x/|v + w|_{max}$. Computations at the interior grid points are performed twice, first using the Explicit-Implicit method [1], and then the Lax–Wendroff two-step scheme.

4. Lax-Wendroff two-step scheme

The Lax–Wendroff two-step scheme is an explicit finite-difference scheme of second-order accuracy. The first step, which may be considered as an intermediate step, is of first-order accuracy. Details describing the method and discussion on stability criteria are available elsewhere [11, 12, 13].

We consider the set of conservation equations (2.1–2.3) to be of the form

$$\frac{\partial Q_{i1}}{\partial t} + \frac{\partial Q_{i2}}{\partial x} = Q_{i3} \tag{4.1}$$

in which $i = 1, 2, 3$. The quantity Q_{i1}, with $i = 1, 2, 3$, represents functions of the dependent variables P, α and v. Applying the Lax–Wendroff two-step finite-difference scheme to equation (4.1) we obtain:

First Step:

$$Q_{i1}(x + \Delta x, t + \Delta t) = 0.5[Q_{i1}(x + 2\Delta x, t) + Q_{i1}(x, t)]$$
$$- 0.5p[Q_{i2}(x + 2\Delta x, t) - Q_{i2}(x, t)]$$
$$+ 0.5\Delta t[Q_{i3}(x + 2\Delta x, t) + Q_{i3}(x, t)] + O(\Delta x^2, \Delta t), \qquad (4.2)$$

Second Step:

$$Q_{i1}(x, t + 2\Delta t) = Q_{i1}(x, t) - p[Q_{i2}(x + \Delta x, t + \Delta t) - Q_{i2}(x - \Delta x, t + \Delta t)]$$
$$+ \Delta t[Q_{i3}(x + \Delta x, t + \Delta t) + Q_{i3}(x - \Delta x, t + \Delta t)] + O(\Delta x^2, \Delta t^2). \qquad (4.3)$$

Equations (4.2) and (4.3) are used for the computational steps shown schematically in Figure 2. The Courant–Friedrichs–Lewy stability condition is satisfied.

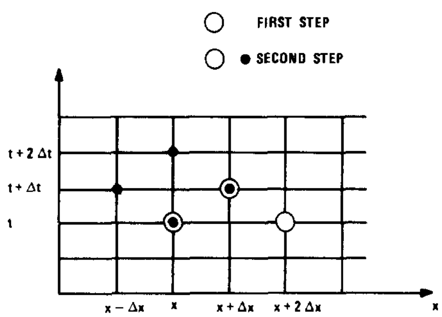

Figure 2. Definition sketch of two-step Lax–Wendroff procedure in $x - t$ plane.

When a discontinuity such as a shock wave develops the Lax–Wendroff scheme produces an overshooting of the shock front, followed by oscillations. These numerical oscillations have to be damped by introducing additional dissipation. Based on Shuman's filtering operator method, as described by Vliegenthart [8], Kranenburg [4] has shown, for a similar problem of wave propagation in bubbly mixtures, that smoothing of the oscillations can be accomplished without significantly affecting the low frequency waves associated with the transient. This method is used here to smooth the numerical oscillations following the discontinuity. A smoothing parameter is defined as

$$\theta_i(x, t) = \{0.5Q_{i1}(x + 2\Delta x, t) - Q_{i1}(x, t) + 0.5Q_{i1}(x - 2\Delta x, t)\}/Q_{i\gamma} \qquad (4.4)$$

where $Q_{i\gamma}$ is a reference interval variable of $Q_{i1}(x, t)$. If θ_i exceeds a reference value θ_γ, numerical viscosity is added as follows. If $|\theta_i| > \theta_\gamma$:

$$\bar{Q}_{i1}(x, t) = Q_{i1}(x, t) + 0.5Q_{i\gamma}\theta_i(x, t) \qquad (4.5)$$

where $\bar{Q}_{i1}$ is the corrected value of Q_{i1}. It is necessary to ascertain the effect of θ_γ and to select a value, by trial, that yields just sufficient smoothing. Figure 3 illustrates the effect of θ_γ and shows that a value of $\theta_\gamma = 0.03$ appears to yield satisfactory results. Kranenburg [4]

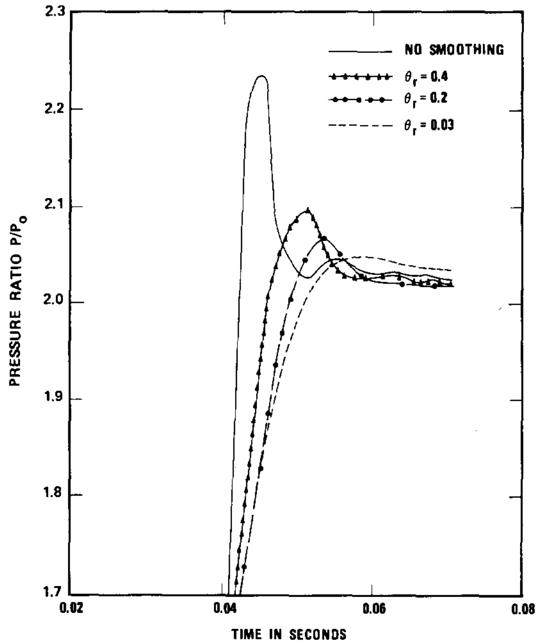

Figure 3. Effect of smoothing on Lax–Wendroff scheme.

shows that, if $\theta_\gamma \geq 8\pi^2/(L/\Delta x)^2$, L being the length of the pipe, the smoothing procedure does not influence low-frequency transients. This criterion proved useful in establishing limits on θ_γ.

Using the $L - W$ scheme the pipe was divided into 120 divisions ($N = 120$) and Δt was evaluated at each time step in order to satisfy the CFL condition. For values of the mesh ratio p ranging from 0.0027 to 0.0045 the computations were carried out for 1.0 sec. The computed values at every second step (second-order accuracy) were used to obtain the pressure traces shown in Figure 7.

5. Explicit-Implicit method [1]

The Explicit-Implicit scheme formulated by McGuire and Morris [1] is a combination of the explicit scheme [9] and the implicit scheme [10, 12] they proposed earlier. We consider the system of equations represented by equation (4.1) and denote by $(j\Delta x, m\Delta t)$ the nodal points of the mesh, where j ranges over the integers $0, 1, 2, \ldots, N$ and m assumes integer values $1, 2, \ldots$ etc. N is the total number of divisions given by $L/\Delta x$. The approximations to $Q_{i1}[j\Delta x, (m+1)\Delta t]$, based on the class of explicit schemes introduced by McGuire and Morris [9], are obtained from the following set of finite-difference equations:

$$Q_{i1}^*[(j\Delta x, (m+a)\Delta t)] = \tfrac{1}{2}\{Q_{i1}[(j+\tfrac{1}{2})\Delta x, m\Delta t] + Q_{i1}[(j-\tfrac{1}{2})\Delta x, m\Delta t]\}$$
$$- ap\{Q_{i2}[(j+\tfrac{1}{2})\Delta x, m\Delta t] - Q_{i2}[(j-\tfrac{1}{2})\Delta x, m\Delta t]\}$$
$$+ \frac{a\Delta t}{2} Q_{i3}[(j+\tfrac{1}{2})\Delta x, m\Delta t] + Q_{i3}[(j-\tfrac{1}{2})\Delta x, m\Delta t] \tag{5.1}$$

$$Q_{i1}[j\Delta x, (m+1)\Delta t] = Q_{i1}[j\Delta x, m\Delta t]$$
$$-\frac{p}{2}\left(1-\frac{1}{2a}\right)\{Q_{i2}[(j+1)\Delta x, m\Delta t] - Q_{i2}[(j-1)\Delta x, m\Delta t]\}$$
$$+\frac{p}{2a}\{Q^*_{i2}[(j+\tfrac{1}{2})\Delta x, (m+a)\Delta t] - Q^*_{i2}[(j-\tfrac{1}{2})\Delta x, (m+a)\Delta t]\}$$
$$+\frac{\Delta t}{2}\{Q_{i3}[(j+1)\Delta x, m\Delta t] + Q_{i3}[(j-1)\Delta x, m\Delta t]\}. \tag{5.2}$$

In equations (5.1) and (5.2) Q^* represents a first-order approximation to Q. The explicit scheme given by these equations is second-order accurate and stable if the CFL condition is satisfied.

The class of implicit methods given by McGuire and Morris consists of equations (5.1) and an equation for $Q_{i1}[j\Delta x, (m+1)\Delta t]$, as follows:

$$Q_{i1}[j\Delta x, (m+1)\Delta t] = Q_{i1}[j\Delta x, m\Delta t]$$
$$-\frac{p}{2}[\tfrac{1}{2} + d(a-1)]\{Q_{i2}[(j+1)\Delta x, m\Delta t] - Q_{i2}[(j-1)\Delta x, m\Delta t]\}$$
$$+\frac{p}{2}(\tfrac{1}{2} - ad)\{Q_{i2}[(j+1)\Delta x, (m+1)\Delta t] - Q_{i2}[(j-1)\Delta x, (m+1)\Delta t]\}$$
$$+ pd\{Q^*_{i2}[(j+\tfrac{1}{2})\Delta x, (m+a)\Delta t] - Q^*_{i2}[(j-\tfrac{1}{2})\Delta x, (m+a)\Delta t]\}$$
$$+\frac{\Delta t}{2}\{Q_{i3}[(j+1)\Delta x, m\Delta t] + Q_{i3}[(j-1)\Delta x, m\Delta t]\}. \tag{5.3}$$

This scheme is shown to be second-order accurate and stable if $ad > 0$ and $p|\lambda| \le 1/\sqrt{2ad}$, where $|\lambda|$ is the maximum modulus eigenvalue of the coefficient matrix of the linearized version of the system (4.1).

The Explicit-Implicit scheme of McGuire and Morris [1] is a combination of equations (5.1) and (5.2) at grid points with $(m+i)$ odd, and equations (5.1) and (5.3) at other grid points as shown in Figure 4. The scheme is of second-order accuracy and has been shown to be stable if and only if $p|\lambda| \le 1$ and $-\frac{1}{2} \le ad \le \frac{1}{2}$.

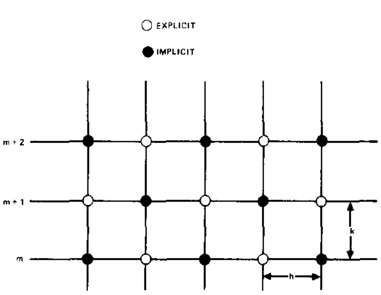

Figure 4. Definition sketch for Explicit-Implicit computational schemes.

For the computations of the boundary-value problem, discussed earlier, the pipe length was divided into 60 divisions. This means that the time interval Δt at which second-order accuracy was obtained is comparable with that for the Lax–Wendroff scheme. This is, of course, necessary for a comparison of results. It should be noted that the CFL condition is to be satisfied for both Lax–Wendroff and Explicit-Implicit schemes and hence the selection of Δt is done the same way.

In the Explicit-Implicit scheme the overshooting and the oscillations associated with the computed shock profile can be minimized, if not eliminated completely, by a proper selection of a and d. Stability requirements suggest $ad \leq \frac{1}{2}$. Various positive values of a and d have been tried to ascertain a most suitable combination of a and d, as illustrated in Figures 5 and 6. It was observed that when $ad > \frac{1}{2}$, numerical instability did occur. McGuire and Morris [1] have shown that, for values of a and d which are fairly large and of about the same magnitude, reasonable profiles could be obtained. But the values of the mesh size, p used in their examples are much larger than those encountered in the present problem. The value of p for the present study ranges from 0.0057 to 0.0090. As $a = 2$ and $d = \frac{1}{4}$ appear to give fairly good results (see Figure 6) they were used for all computations. The transient pressure computed with the Explicit-Implicit method is also presented in Figure 7 together with that found using the Lax–Wendroff scheme.

6. Comparison of computed results

The Explicit-Implicit scheme requires only about 88 percent of the computing time required by the Lax–Wendroff two-step scheme. Upon comparing the two pressure records of Figure 7 it is seen that the computed transient pressures using the Explicit-Implicit Scheme agree well with those using the $L-W$ two-step scheme. The Explicit-Implicit scheme has an added advantage in that the numerical oscillations behind the shock waves can be reduced by a suitable selection of the parameters a and d. The shock front is seen to be steeper and better simulated (Figure 7) by the Explicit-Implicit scheme than with the $L-W$ scheme, where the use of a smoothing method has caused a higher spreading of the front. Furthermore, the Explicit-Implicit method incorporates the boundary conditions at each step for the calculations of the variables at that particular time. This is not so with the $L-W$ scheme, since it is completely explicit.

7. Conclusions

The use of the Explicit-Implicit method [1] for solving an initial value-boundary value problem of transient two-phase flow for bubbly flowing mixtures in a conduit has been demonstrated. In the event of a discontinuity due to shock-wave formation, the direct numerical integration using the Explicit-Implicit method [1] is shown to be much simpler than the commonly used method of characteristics, which in this case involves a shock-fitting procedure. The Explicit-Implicit method [1] is simple and versatile, and because of the implicit set of computations is more suitable for boundary-value problems than such explicit methods as the Lax–Wendroff scheme. The computations along the boundary can be accomplished using the method of characteristics.

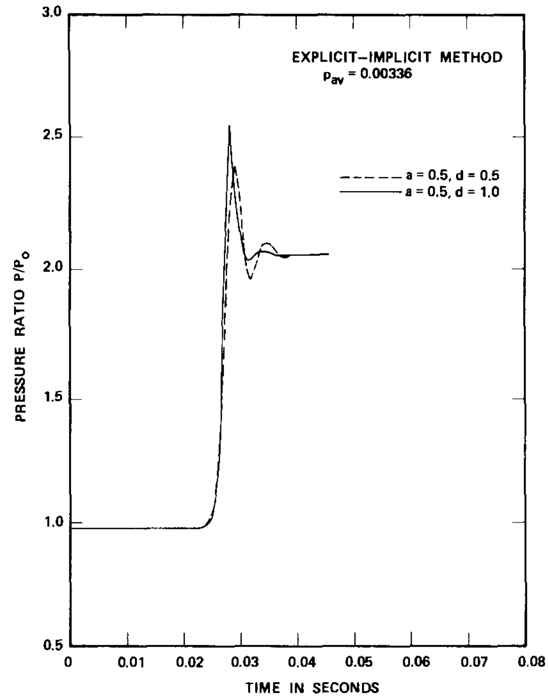

Figure 5. Influence of parameters a and d on smoothing of oscillations behind shock. $P_0 = 0.327$ MPa, $\alpha_0 = 0.0162$, $v_0 = 1.52$ m/sec, and $x/L = 0.75$.

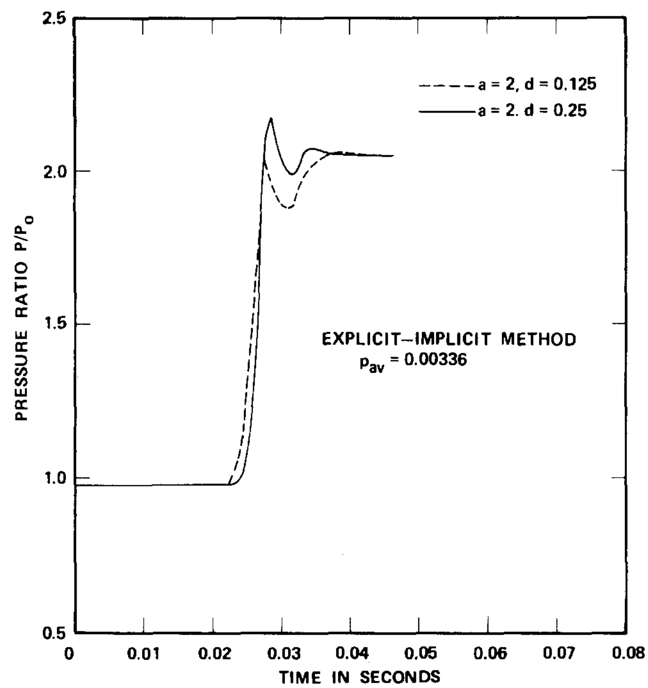

Figure 6. Influence of parameters a and d on smoothing of oscillations behind shock. $P_0 = 0.327$ MPa, $\alpha_0 = 0.162$, $v_0 = 1.52$ m/sec, and $x/L = 0.75$.

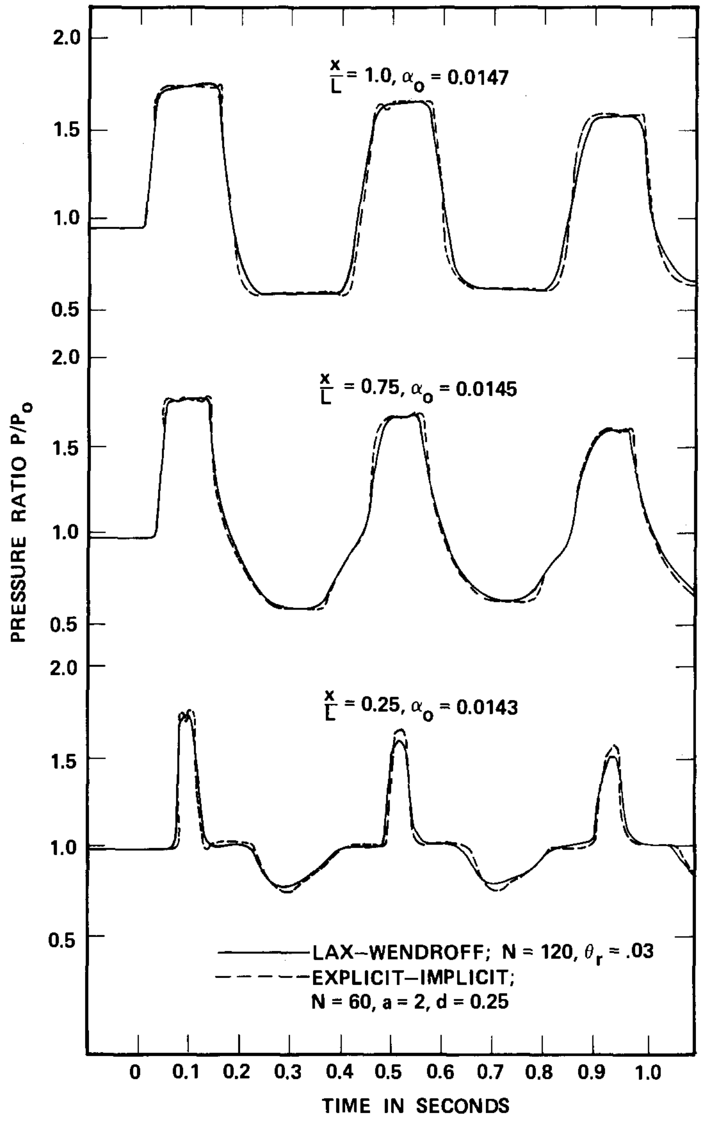

Figure 7. Comparison of Explicit-Implicit scheme with Lax–Wendroff method. $P_0 = 0.38$ MPa, $v_0 = 1.5$ m/sec.

Based on the computational procedure used for the particular problem discussed here, the Explicit-Implicit method requires only 88% of the computation time needed with Lax–Wendroff two-step scheme. Furthermore, with the Explicit-Implicit method the smoothing of numerical oscillations following a discontinuity can be accomplished conveniently by choosing proper values of the parameters a and d.

The Explicit-Implicit method is also seen to simulate the shock front better than the Lax–Wendroff scheme, for which more spreading of the discontinuity is noted with the use of the smoothing procedure.

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